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RESEARCH MEMORANDUM

STABILITY OF SYSTEMS CONTAINING A HEAT

SOURCE - THE RAYLEIGH CRITERION

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SUMMARY

The stability of systems containing a heat source is examined from the energy point of view. Rayleigh's criterion is derived. In the case of a flame, it is found that Rayleigh's criterion must be modified slightly if the specific-heat ratios γ of the burned and unburned gases are different.

INTRODUCTION

It has long been known that oscillatory phenomena are of common occurrence in systems containing an energy source, whether it is distributed in space or concentrated in a limited region (e.g., in a plane or a surface). Some typical examples are the screaming and chugging of afterburners and combustion chambers, as well as the more classical examples, such as the Rijke tone, the singing flame, and the oscillatory flame propagation in a tube. Generally speaking, the phenomena possess two characteristics. First, an oscillation is built up without any appreciable external excitation. Secondly, after the oscillation has increased to a certain amplitude, it is maintained in this state (unless the system breaks down before this state is attained).

A dynamical system will start to oscillate with increasing amplitude only if energy is fed into the system in such a way that there is a net increase of the total mechanical energy of the system after each cycle of oscillation. The vibration is finally maintained at a given level when the mechanical energy fed into the system per cycle of oscillation is just equal to the sum of that dissipated by viscosity and that radiated away from the system per cycle of oscillation. When the amount of energy released per second is controlled by an external agency and is independent of the fluctuation inside the system, the oscillation will build up when the energy is released at certain characteristic frequencies. Such a phenomenon is usually described as resonance. On the other hand, if the system itself contains an energy source with the property that the amount of energy release depends upon the fluctuation inside the system, an accidental small disturbance inside the system may interact with the energy source in such a manner that mechanical energy will be fed into the disturbance per cycle of oscillation, building up its amplitude to an appreciable extent. Depending on the nature of the response of the energy source to the disturbance, the interaction of the two may lead to a buildup or damping of the oscillation in the system. When the oscillation does build up, the phenomenon is usually described as instability. It is the purpose of this analysis to study the conditions under which systems containing heat sources may be expected to exhibit instability.

To illustrate more precisely what has been described and also as an introduction to the problems proposed for study, consider a special example. Consider a tube of length L with open ends. Let the origin of the coordinate system be chosen at one end of the tube with the x-axis parallel to the wall of the tube. Let $Q(x,t)$ be the rate of heat release to the medium per unit volume per unit time at the point x and at the instant t. Let c_p be the specific heat at constant pressure of the gas in the tube and T_o be the temperature of the gas when $Q = 0$. If Q/c_pT_o and its time derivative are "small," the pressure fluctuation in the tube will also be small and is governed by the equation

$$\frac{1}{c^2}\frac{\partial^2 P}{\partial t^2} - \frac{\partial^2 P}{\partial x^2} = \frac{\partial}{\partial t}\left(\frac{Q}{c_pT_o}\right) \tag{1a}$$

where $P = \delta p$ is the pressure fluctuation (above the mean pressure p_o), γ is the specific-heat ratio, and c is the velocity of sound (see, e.g., ref. 1). At the two ends of the tube, there exists the condition

$$P = 0 \qquad (x = 0,\ x = L) \tag{1b}$$

for all values of t. If $Q(x,t)$ is a given function of x and t, the nonhomogeneous term at the right of the equation is known and represents the "forcing function" or "source distribution." The problem is then one of forced oscillation. If the initial conditions are specified, for example,

$$\left.\begin{aligned} (P)_{t=0} &= f_1(x) \\ \left(\frac{\partial P}{\partial t}\right)_{t=0} &= f_2(x) \end{aligned}\right\} \tag{1c}$$

a solution can be constructed readily. It can be easily shown that when $Q(x,t)$ is periodic in t with a period commeasurable with the natural period of the tube, that is, $2L/c$, the amplitude of a mode of oscillation will increase linearly with time. This is the resonance phenomenon. Now, if the rate of heat release Q is not a prescribed function of x and t but is given in terms of the fluctuations in the system, for example, Q is directly proportional to the pressure fluctuation P, the differential equation governing the pressure field becomes a homogeneous equation. Thus, if

$$\frac{Q}{c_p T_o} = kP \tag{2a}$$

then equation (1a) becomes

$$\frac{1}{c^2}\frac{\partial^2 P}{\partial t^2} - \frac{\partial^2 P}{\partial x^2} = k\frac{\partial P}{\partial t} \tag{2b}$$

The solution of this equation satisfying boundary condition (1b) and initial conditions (1c) is

$$P = e^{\frac{c^2 kt}{2}} \sum_{n=1}^{\infty} (A_n \cos \omega_n t + B_n \sin \omega_n t) \sin \frac{n\pi x}{L} \tag{2c}$$

where

$$\omega_n = \sqrt{\left(\frac{n\pi c}{L}\right)^2 - \frac{k^2 c^4}{4}} \tag{2d}$$

$$A_n = \frac{2}{L}\int_0^L f_1(x) \sin \frac{n\pi x}{L}\, dx \tag{2e}$$

$$B_n = \frac{2}{L\omega_n}\int_0^L \left[f_2(x) - \frac{kc^2}{2} f_1(x)\right] \sin \frac{n\pi x}{L}\, dx \tag{2f}$$

From equation (2c) it is seen that, if $k > 0$, the initial disturbances will grow exponentially with the time t and such a system is unstable. If $k \leqq 0$, the solution clearly shows that the system is stable to small disturbances.

The same conclusion can be arrived at more simply by using the formal method of stability analysis. As usual a disturbance of the form $P = e^{i\beta t}\psi(x)$ is assumed. To satisfy the differential equation it is necessary to have

$$\psi(x) = \tilde{A} \cos \lambda(\beta)x + \tilde{B} \sin \lambda(\beta)x \tag{2g}$$

where $\tilde{A}$ and $\tilde{B}$ are constants and

$$\lambda(\beta) = \sqrt{\frac{\beta^2}{c^2} + ik\beta} \tag{2h}$$

To satisfy the boundary condition (1b), it is necessary that $\tilde{A} = 0$ and $\lambda(\beta) = \frac{n\pi}{L}$, n being a positive integer. Solving for β from the second condition gives

$$\begin{aligned} \beta_n &= \frac{c^2}{2}\left[-ik \pm \sqrt{4\left(\frac{n\pi}{L}\right)^2 - k^2}\right] \\ &= -i\frac{kc^2}{2} \pm \omega_n \end{aligned} \tag{2j}$$

Consequently, if $k > 0$, both roots have the property that $\text{Im}\ \beta_n < 0$; if $k = 0$, $\text{Im}\ \beta_n = 0$; and if $k < 0$, both roots are such that $\text{Im}\ \beta_n > 0$

where Im indicates imaginary roots. Since $P = e^{i\beta t}\psi(x)$, the system is unstable for $k > 0$, but stable for $k \leqq 0$. In this formal analysis, usually no time is taken to satisfy the initial conditions (1c) in the belief that these initial conditions can always be satisfied by the superposition of the assumed form of disturbances. In the present analysis, this is indeed the case since the eigenfunctions $\psi(x)$ corresponding to $n = 1, 2, \ldots$ form a complete orthonormal set in the interval $0 \leqq x \leqq L$.

In the above example, when the system is unstable (i.e., $k > 0$), the amplitude of the fluctuation grows exponentially with time without ultimately settling down into a state of permanent oscillation of constant amplitude (i.e., into a limit cycle). If, however, the rate of heat release Q is related to the pressure fluctuation by

$$\frac{Q}{c_p T_o} = \epsilon\left(P P_o^2 - \frac{1}{3} P^3\right) \tag{3a}$$

where ϵ and P_o are two positive quantities, instead of by equation (2a), the differential equation governing the pressure fluctuation becomes

$$\frac{1}{c^2}\frac{\partial^2 P}{\partial t^2} - \frac{\partial^2 P}{\partial x^2} = \epsilon\left(P_o^2 - P^2\right)\frac{\partial P}{\partial t} \tag{3b}$$

Any small disturbances will at first be amplified exponentially and will ultimately settle down into a limit cycle since $\epsilon(P_o^2 - P^2)$ is positive for $|P| < P_o$ and becomes negative for $|P| > P_o$. This is a case of selfsustained vibration of large amplitude. This particular system clearly corresponds to the limit of a very large number of identical Van der Pol oscillators linearly coupled together to form a continuum. It may be of some basic interest to study more closely a nonlinear system like equation (3b).

In the above examples, heat is released at all points in the tube. If heat is released in a portion of the tube or concentrated in a narrow region, similar analysis applies. However, the calculation is, as a rule, longer and the simplest method of attack is probably the formal stability analysis. However, with each slight change in the geometrical arrangement, for example, the extent and connectivity of the region in which heat is released, a new calculation must be performed. In this sense, the above analytic approach is not very satisfactory although it does provide quantitative information as to the rate of amplification of the disturbance. For many purposes, however, precise quantitative information is not so important as having a qualitative idea as to whether a system may or may not be stable and why it is so. The latter is indeed the more important if the fundamental physical principles involved are to be understood. It is toward this goal that the study in the remainder of this paper is directed.

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SYMBOLS

A	cross-sectional area of tube
$\tilde{A}$	constant
A_f	flame area
A_n	defined by equation (2e)
$\tilde{B}$	constant
B_n	defined by equation (2f)
c_p	specific heat at constant pressure
c	velocity of sound
E	total energy, Kinetic energy + Potential energy
$f_1(x),f_2(x)$	arbitrary functions
h	location of heater
k	proportionality constant
L	length of tube
n	positive integer
p	pressure fluctuation above mean level

P_o	positive quantity
p	pressure
$Q,Q(x,t)$	heat release to medium per unit volume per unit time
$\bar{Q}$	heating value of mixture
R	gas constant
S_a	apparent flame speed
S_t	flame speed
T	temperature
T_o	temperature when $Q = 0$
t	time
u	velocity
x	space coordinate
β,β_n	parameter
γ	specific-heat ratio
Δ	increment of total energy after one cycle of oscillation
$\delta(\ )$	increment of ( )
ϵ	position quantity
$\lambda(\beta)$	defined by equation (2h)
ρ	density
$\psi(x)$	eigenmode
ω	rate of heat release per unit cross-sectional area of tube
ω_o	value of ω when there is no disturbance
ω_n	defined by equation (2d)

Subscripts:

1	conditions upstream of heat source
2	conditions downstream of heat source
h	conditions at heat source
i	conditions at inlet end of tube

RAYLEIGH'S CRITERION

In explaining the process of maintenance of vibration by heat, Rayleigh (ref. 2, pp. 226 and 227) states that if heat is periodically communicated to and abstracted from a mass of air vibrating in a cylinder bounded by a piston, for example, the effect produced will depend upon the phase of vibration at which the transfer of heat takes place. If heat is given to the air at the moment of greatest condensation or is taken from it at the moment of greatest rarefaction the vibration is encouraged. On the other hand, if heat is given at the moment of greatest rarefaction or abstracted at the greatest condensation the vibration is discouraged; however, there is no effect in encouraging or discouraging the vibration if the air concerned is at a loop, that is, a place where the density does not vary, or if the communication of heat is the same at any stage of rarefaction as at the corresponding stage of condensation. Rayleigh went on in applying this criterion to explain the singing flame and Rijke's tone. He was able to give a very satisfactory account of the observed effect of tube length on the excitation of singing flame and the effect of the position of the heater in the Rijke's tube in the production of Rijke tone. (See pp. 229 to 230 and 232 to 233 of ref. 2.)

In most later investigations on the maintenance of vibration by heat, Rayleigh's criterion has been quoted (e.g., refs. 3 to 7). Rayleigh himself did not show how he arrived at such an apparently rather general statement. Noting this deficiency, Putnam and Dennis proposed a proof which many find hard to believe (see appendix of ref. 3). Their proof is not wrong, but it contains so many plausible hypotheses of a mathematical nature that one is not sure if the physical conclusion it purports to prove is not more plausible than the mathematical hypotheses used. However, Putnam and Dennis did put Rayleigh's criterion in a very precise form: Mathematically, and neglecting damping forces, Rayleigh's criterion requires that the time integral, over a cycle, of the instantaneous product of the rate of heat release and of the oscillating component of the pressure be greater than zero (ref. 3).[1]

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[1]The criterion in this form was also independently suggested by Bailey (see p. III.9 of ref. 6).

An explanation of Rayleigh's criterion on a physical basis will make it quite clear why the criterion is in fact a very plausible one.[2] For convenience of explanation, consider a tube of finite length. Divide the tube into many fictitious compartments (say of equal size) in one of which heat is being added periodically from an external source. This is illustrated in figure 1 where the compartments are separated by planes (shown as dashed lines in the figure), while the shaded compartment bounded by the planes marked A and A' is the one in which heat is being added. In the absence of viscosity and heat conductivity, there is no loss of generality by replacing the fictitious planes A and A' by two solid plane surfaces (or "pistons") provided that these walls always move in synchronism with the motion of the fictitious planes A and A'. The other compartments serve, as is well known, as a mechanical spring-mass system (or flywheels in the engine analogy) where mechanical energy produced by the "engine" compartment can be stored. The work done by the engine can be easily calculated from the PV-diagram. Depending on when heat is added and subtracted during each cycle of vibration the amount of mechanical work produced is greater than, equal to, or less than zero; this determines whether mechanical energy is given to or extracted from the spring-mass system (i.e., the flywheels). For the case when the amount of heat energy released by the heat source depends on the magnitude of fluctuation in the system and is zero when there is absolutely no fluctuation in the system, the initial disturbances act as an "engine starter." It is at once clear from the PV-diagram that if heat is added when the pressure in the engine is the highest and taken away when the pressure in the engine is the lowest, the maximum amount of mechanical work is obtained. This is in fact identical to Rayleigh's statement that if heat is given to the air at the moment of greatest condensation or is taken from it at the moment of greatest rarefaction the vibration is encouraged. Furthermore, it is also clear from the engine analogy that the real criterion of amplification of a disturbance is that the net mechanical work done by the engine per cycle must be greater than the loss through viscous dissipation and, hence, must at least be greater than zero. This is also in accordance with Rayleigh's remark that there is no effect (of encouraging or discouraging the vibration) if the air concerned is at a loop, that is, a place where the density does not vary, or if the communication of heat is the same at any stage of rarefaction as at the corresponding stage of condensation.

It is possible to put the above criterion in a mathematical form. The result of this calculation leads to the formulation by Putnam and Dennis of Rayleigh's criterion. Since this calculation will appear as a particular case of the analysis of a more useful system discussed in the next section, the details will not be given here.

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[2]The author is indebted to Prof. Leslie S. G. Kovásznay for this explanation.

In the example just presented, heat must be added as well as subtracted from the "engine compartment" if the state of the gas inside the compartment is to be brought back to its initial state after each cycle. The heat subtracted is, of course, less than the heat added if the engine produces a net positive amount of mechanical work every cycle. The heat subtracted corresponds to the heat rejected in an actual engine. To be sure, for most practical cases, the gas in the engine compartment instead of returning to its initial state at the end of a cycle ends up at a temperature slightly higher than its original temperature. This cuts down the mechanical output of the engine in the next cycle and sometimes also reduces the amount of heat given to the engine (e.g., when the rate of heat transfer depends on the temperature of the medium as in the case of conductive heat transfer). These are precisely some of the factors which will eventually limit the amplitude of the oscillation to finite size and account for the selfsustained vibration (i.e., limit cycle) which is often observed in experiments on such systems.

When heat is not added to the gas in the limited region as shown in figure 1 but is distributed throughout the tube, the same argument applies except that it is necessary to think in terms of a multicylinder engine rather than a single-cylinder engine doing work on a spring-mass system. In fact, when the amplitude of the oscillations is small this case can always be analyzed by a superposition of the previous case.

The only drawback of the engine analogy is that it cannot be extended simply to the case when there is a constant current of air through the tube and when the heat added to the medium fluctuates with the disturbance in the system above some nonzero mean value. It is clear that the heat added to the system can be decomposed into two parts, a steady component (i.e., a direct-current component) which maintains a given temperature distribution inside the system and a varying component (i.e., an alternating-current component) which feeds energy into the disturbance. The heat energy that has been given to the disturbance appears in two forms: The pressure waves which carry the mechanical energy into the rest of the system, and the entropy spots which are carried away by the draft and represent the amount of energy going into the heating of the gas above the mean temperature distribution. The latter is therefore unavailable for doing mechanical work. In such a case, the entropy spots carried away by the draft will be the counterpart of the heat energy rejected by the engine. It is possible to study the energy transformation and, in particular, the total amount of the mechanical energy stored in the system per cycle of vibration from a purely analytical standpoint (i.e., without appealing to a physical model). This will be done in the following sections. The physical model, however, is useful in interpreting the analytical formulation and conclusions.

ANALYTICAL FORMULATION

It is the purpose of the present analysis to establish the condition under which a small disturbance is amplified in the course of time by its interaction with a heat source. Though the analysis is restricted to small disturbances for analytical reasons, the derived condition is nevertheless useful in practice since small disturbances always exist in any physical setup and also because in most cases a large disturbance does not disappear without first becoming small.

For definiteness, consider a gaseous medium in a tube of length L. The origin of the coordinate axis is chosen at one end of the tube and the x-axis, parallel to the wall of the tube. The two ends of the tube will then be given by $x = 0$ and $x = L$. The heat source is assumed to be concentrated in a single plane at $x = h$.[3] It is further assumed that there is a constant current of air through the tube flowing into the tube from the end $x = 0$ with a speed much smaller than the local speed of sound. Finally, it is assumed that in the absence of any disturbance the heat source is releasing energy at a rate of ω_o units of heat per second per unit of cross-sectional area. As a result of the heat addition, the state of flow ahead of and behind the heater will not be the same. Let p_1, ρ_1, T_1, and u_1 denote, respectively, the pressure, density, temperature, and velocity of the undisturbed medium upstream of the heat source (i.e., for $0 < x < h$), while p_2, ρ_2, T_2, and u_2 denote, respectively, those downstream of the heat source (i.e., for $h < x < L$; see fig. 2). These quantities are related by the continuity, momentum, and energy equations across the heater which are

$$\rho_1 u_1 = \rho_2 u_2 \tag{4a}$$

$$p_1 + \rho_1 u_1^2 = p_2 + \rho_2 u_2^2 \tag{4b}$$

$$\omega_o = \rho_2 u_2\left(c_{p_2} T_2 + \frac{1}{2} u_2^2\right) - \rho_1 u_1\left(c_{p_1} T_1 + \frac{1}{2} u_1^2\right) \tag{4c}$$

In case the heat source is a flame front which propagates with a flame speed S_t relative to and against the oncoming stream, the heat source

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[3]This assumption by no means limits the generality of the results obtained below since, for small disturbances, a continuously distributed heat source can always be regarded as a superposition of a large number of concentrated heat sources.

will be seen to have an apparent velocity S_a (taken as positive if the heat source moves forward against the oncoming stream) given by

$$S_a = S_t - u_1 \tag{5a}$$

In such case, equations (4) must be replaced by the following equations:

$$\rho_1\left(u_1 + S_a\right) = \rho_2\left(u_2 + S_a\right) \tag{5b}$$

$$p_1 + \rho_1\left(u_1 + S_a\right)^2 = p_2 + \rho_2\left(u_2 + S_a\right)^2 \tag{5c}$$

$$\omega_o = \rho_2\left(u_2 + S_a\right)\left[c_{p_2}T_2 + \frac{1}{2}\left(u_2 + S_a\right)^2\right] - \rho_1\left(u_1 + S_a\right)\left[c_{p_1}T_1 + \frac{1}{2}\left(u_1 + S_a\right)^2\right] \tag{5d}$$

Evidently, the system given by equations (4) can be considered as a particular case of the system given by equations (5); for, if $S_t = u_1$ in equation (5a) it is found that $S_a = 0$ and equations (5) reduce to equations (4). Consequently, only system (5) needs to be considered.

If p_1, ρ_1, T_1, u_1, ω_o, and S_t are assumed to be given, then equations (5a) to (5d), together with the gas law

$$p_2 = \rho_2 R_2 T_2 \tag{5e}$$

will enable calculation of the five unknowns p_2, ρ_2, T_2, u_2, and S_a. In particular, it can be easily verified that $\left(u_2 + S_a\right)\Big/\sqrt{\gamma_2 R_2 T_2}$ is of the same order of magnitude as $\left(u_1 + S_a\right)\Big/\sqrt{\gamma_1 R_1 T_1}$. Consequently, if the velocity of the direct current through the tube and S_t are small compared with the local sound speed (so that terms which are of the order of the square of the Mach numbers can be dropped), equations (5) can be reduced to

$$\rho_1\left(u_1 + S_a\right) = \rho_2\left(u_2 + S_a\right) \tag{6a}$$

$$p_1 = p_2 \tag{6b}$$

$$\omega_0 = \frac{\gamma_2}{\gamma_2 - 1} p_2\left(u_2 + S_a\right) - \frac{\gamma_1}{\gamma_1 - 1} p_1\left(u_1 + S_a\right) \tag{6c}$$

where

$$S_t = u_1 + S_a \tag{6d}$$

$$p_2 = \rho_2 R_2 T_2 \tag{6e}$$

Now, suppose that there are some disturbances inside the tube. Let the pressure, density, temperature, and velocity of the medium upstream of the heat source at the point x and instant t be denoted by $p_1 + \delta p_1$, $\rho_1 + \delta\rho_1$, $T_1 + \delta T_1$, and $u_1 + \delta u_1$ while those behind the heat source are denoted by $p_2 + \delta p_2$, $\rho_2 + \delta\rho_2$, $T_2 + \delta T_2$, and $u_2 + \delta u_2$. If the disturbances are weak $\left(\text{i.e., if } \frac{\delta p_1}{p_1}, \frac{\delta\rho_1}{\rho_1}, \frac{\delta T_1}{T_1}, \frac{\delta u_1}{c_1}, \frac{\delta p_2}{p_2}, \frac{\delta\rho_2}{\rho_2}, \frac{\delta T_2}{T_2}, \text{ and } \frac{\delta u_2}{c_2} \ll 1, \text{ where } c \text{ denotes the velocity of sound}\right)$, the energy in the disturbances can be decomposed into two parts: The kinetic energy is given by

$$\frac{1}{2} \rho_1 \int_0^h \left(\delta u_1\right)^2 A\ dx + \frac{1}{2} \rho_2 \int_h^L \left(\delta u_2\right)^2 A\ dx \tag{7a}$$

(A being the cross-sectional area of the tube) and the potential energy is given by

$$\frac{1}{2}\rho_1 c_1^2 \int_0^h \left(\frac{\delta p_1}{\gamma_1 p_1}\right)^2 A\ dx + \frac{1}{2}\rho_2 c_2^2 \int_h^L \left(\frac{\delta p_2}{\gamma_2 p_2}\right)^2 A\ dx \tag{7b}$$

(see, e.g., pp. 17 and 18 of ref. 2). The total energy in the disturbances is the sum of these two equations. If the total energy in the disturbances increases after each cycle of oscillation (described subsequently), it is said that the disturbance is amplified by the heat source. Now, it is a well-known fact that the surface integral of the product of the pressure and the velocity component in the direction of the normal of a control surface is related to the rate of change of energy inside the system. (See, e.g., ref. 8.) This suggests the following calculation.

Consider a control surface shown by dashed lines in figure 3. It consists of two compartments: The first encloses the medium ahead of the heat source, and the second encloses the medium behind the heater. Consider the first compartment. If the subscripts h and i are used to denote the condition of flow at the heat source and at the inlet end of the tube, respectively, the following identities are obtained:

$$\begin{aligned}
\left(p_1 + \delta p_1\right)_h\left(u_1 + \delta u_1\right)_h - \left(p_1 + \delta p_1\right)_i\left(u_1 + \delta u_1\right)_i &= \int_0^h \frac{\partial}{\partial x}\left[\left(p_1 + \delta p_1\right)\left(u_1 + \delta u_1\right)\right] dx \\
&= \int_0^h \left[\left(p_1 + \delta p_1\right)\frac{\partial}{\partial x}\left(u_1 + \delta u_1\right) + \right. \\
&\quad \left.\left(u_1 + \delta u_1\right)\frac{\partial}{\partial x}\left(p_1 + \delta p_1\right)\right] dx \\
&= p_1\left[\left(\delta u_1\right)_h - \left(\delta u_1\right)_i\right] + u_1\left[\left(\delta p_1\right)_h - \left(\delta p_1\right)_i\right] + \\
&\quad \int_0^h \left[\left(\delta p_1\right)\frac{\partial}{\partial x}\left(\delta u_1\right) + \left(\delta u_1\right)\frac{\partial}{\partial x}\left(\delta p_1\right)\right] dx
\end{aligned}$$

that is,

$$\left(\delta p_1\right)_h\left(\delta u_1\right)_h - \left(\delta p_1\right)_i\left(\delta u_1\right)_i = \int_0^h \left[\delta p_1 \frac{\partial}{\partial x}\left(\delta u_1\right) + \delta u_1 \frac{\partial}{\partial x}\left(\delta p_1\right)\right] dx \tag{8a}$$

When the inlet end is open, $\left(\delta p_1\right)_i = 0$; hence,

$$\left(\delta p_1\right)_h\left(\delta u_1\right)_h = \int_0^h \left[\delta p_1 \frac{\partial}{\partial x}\left(\delta u_1\right) + \delta u_1 \frac{\partial}{\partial x}\left(\delta p_1\right)\right] dx \tag{8b}$$

(This formula is still true if the inlet end is closed since $(\delta u_1)_i = 0$.) Likewise, if the compartment of the control surface behind the heater is considered,

$$\left(\delta p_2\right)_h\left(\delta u_2\right)_h = -\int_h^L \left[\left(\delta p_2\right) \frac{\partial}{\partial x}\left(\delta u_2\right) + \left(\delta u_2\right) \frac{\partial}{\partial x}\left(\delta p_2\right)\right] dx \tag{8c}$$

It follows from the last two formulas, equations (8b) and (8c), that

$$\left(\delta p_2\right)_h\left(\delta u_2\right)_h - \left(\delta p_1\right)_h\left(\delta u_1\right)_h = -\int_0^h \left[\delta p_1 \frac{\partial}{\partial x}\left(\delta u_1\right) + \delta u_1 \frac{\partial}{\partial x}\left(\delta p_1\right)\right] dx - \int_h^L \left[\delta p_2 \frac{\partial}{\partial x}\left(\delta u_2\right) + \delta u_2 \frac{\partial}{\partial x}\left(\delta p_2\right)\right] dx \tag{8d}$$

Note that this equation is exact for tubes with open (or closed) ends; that is, no neglection has been made. Next, the simplifying assumption that the disturbances in the system are small is introduced into equation (8d). It will be shown that the right-hand side of the above expression gives precisely the rate of increase of total energy inside the tube if the Mach number of the steady current through the tube is of the order of magnitude of 0.01 or less $\left(\text{e.g., if } u_1 \leqq 10 \text{ fps, and } c_1 = 1{,}000 \text{ fps, then } \frac{u_1}{c_1} \leqq 0.01\right)$. For such low Mach numbers, the hydrodynamic equations governing the disturbance can be written simply as

$$\frac{\partial}{\partial t}\left(\frac{\delta \rho_1}{\rho_1}\right) + c_1 \frac{\partial}{\partial x}\left(\frac{\delta u_1}{c_1}\right) = 0 \tag{9a}$$

$$\frac{\partial}{\partial t}\left(\frac{\delta u_1}{c_1}\right) + c_1 \frac{\partial}{\partial x}\left(\frac{\delta p_1}{\gamma_1 p_1}\right) = 0 \tag{9b}$$

$$\frac{\partial}{\partial t}\left(\frac{\delta \rho_1}{\rho_1}\right) - \frac{\partial}{\partial t}\left(\frac{\delta p_1}{\gamma_1 p_1}\right) = 0 \tag{9c}$$

$$\frac{\delta T_1}{T_1} = \frac{\delta p_1}{p_1} - \frac{\delta \rho_1}{\rho_1} \tag{9d}$$

for the region ahead of the heat source. Likewise, similar equations (with subscript 2 instead of 1) hold for the region behind the heat source.

Consider the first integral on the right-hand side of equation (8d). The following equation is obtained by use of equations (9):

$$- \int_0^h \left[\delta p_1 \frac{\partial}{\partial x}\left(\delta u_1\right) + \delta u_1 \frac{\partial}{\partial x}\left(\delta p_1\right)\right] dx = \int_0^h \left[\delta p_1 \frac{\partial}{\partial t}\left(\frac{\delta p_1}{\gamma_1 p_1}\right) + \frac{\gamma_1 p_1}{{c_1}^2}\left(\delta u_1\right) \frac{\partial}{\partial t}\left(\delta u_1\right)\right] dx$$

$$= \frac{\partial}{\partial t} \int_0^h \left[\frac{1}{2} \frac{\left(\delta p_1\right)^2}{\gamma_1 p_1} + \frac{1}{2} \rho_1 \left(\delta u_1\right)^2\right] dx \tag{10a}$$

and gives the rate of increase of the disturbance energy (per unit cross-sectional area of the tube) in the region ahead of the heat source.

Likewise,

$$-\int_h^L \left[\delta p_2 \frac{\partial}{\partial x}\left(\delta u_2\right) + \delta u_2 \frac{\partial}{\partial x}\left(\delta p_2\right)\right] dx = \frac{\partial}{\partial t}\int_h^L \left[\frac{1}{2}\frac{\left(\delta p_2\right)^2}{\gamma_2 p_2} + \frac{1}{2}\rho_2\left(\delta u_2\right)^2\right] dx \tag{10b}$$

gives the rate of increase of the disturbance energy (per unit cross-sectional area) in the region behind the heat source. Substituting equations (10a) and (10b) into equation (8d) leads to the following conclusion: The rate of increase in the total energy E in the disturbance is given by

$$\frac{\partial}{\partial t}(E) = A\left[\left(\delta p_2\right)_h\left(\delta u_2\right)_h - \left(\delta p_1\right)_h\left(\delta u_1\right)_h\right] \tag{11}$$

Next, the right-hand side of the above equation must be evaluated in terms of the rate of heat release at the heat source. Since equations (6) are the mathematical representations of the conservation laws at the heat source, they must be satisfied at all instants provided that instantaneous values of the pressure, velocity, and other physical variables are used instead of p_1, u_1, and so forth. In particular, equations (6b) and (6c) must be satisfied; that is,

$$p_1 + \delta p_1 = p_2 + \delta p_2 \tag{12a}$$

$$\omega_0 + \delta\omega = \frac{\gamma_2}{\gamma_2 - 1}\left(p_2 + \delta p_2\right)\left(u_2 + S_a + \delta u_2 + \delta S_a\right) -$$

$$\frac{\gamma_1}{\gamma_1 - 1}\left(p_1 + \delta p_1\right)\left(u_1 + S_a + \delta u_1 + \delta S_a\right) \tag{12b}$$

In writing the second equation, allowance has been made for the change in the rate of heat release with the fluctuations at the heat source.

Since these relations are valid only at the heater, the subscript h should have been used to indicate this fact. However, for the sake of simplicity, this subscript will not be attached to the various variables involved until the final result is obtained (i.e., eqs. (13)). Making use of equations (6b) and (6c), the last two equations can be rewritten as

$$\delta p_1 = \delta p_2 \tag{12c}$$

$$\frac{\delta\omega}{p_1 c_1} = \frac{\gamma_2}{\gamma_2 - 1}\,\frac{\delta u_2 + \delta S_a}{c_1} - \frac{\gamma_1}{\gamma_1 - 1}\,\frac{\delta u_1 + \delta S_a}{c_1} +$$

$$\frac{\gamma_2}{\gamma_2 - 1}\,\frac{\delta p_2}{p_2}\,\frac{\left(u_2 + S_a\right) + \left(\delta u_2 + \delta S_a\right)}{c_1} -$$

$$\frac{\gamma_1}{\gamma_1 - 1}\,\frac{\delta p_1}{p_1}\,\frac{\left(u_1 + S_a\right) + \left(\delta u_1 + \delta S_a\right)}{c_1} \tag{12d}$$

Equation (12b) has been put in a nondimensional form so that the order of magnitude of the various terms can be readily compared. If terms which are products of the small quantities $\frac{\delta p_1}{p_1}$, $\frac{\delta p_2}{p_2}$, $\frac{\delta u_1}{c_1}$, $\frac{\delta u_2}{c_2}$, $\frac{u_1}{c_1}$, and $\frac{u_2}{c_2}$ are dropped,[4] equation (12d) becomes

$$\frac{\delta\omega}{p_1 c_1} = \frac{\gamma_2}{\gamma_2 - 1}\,\frac{\delta u_2 + \delta S_a}{c_1} - \frac{\gamma_1}{\gamma_1 - 1}\,\frac{\delta u_1 + \delta S_a}{c_1} \tag{12e}$$

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[4]Note that this is the same assumption which was used in deriving equations (9).

Introducing now the subscript h to indicate that these relations are valid only at the heat source, equations (12c) and (12e) can be rewritten in the form

$$\left(\delta p_2\right)_h = \left(\delta p_1\right)_h \tag{13a}$$

$$\left(\delta u_2\right)_h - \left(\delta u_1\right)_h = \frac{\gamma_2 - 1}{\gamma_2}\frac{\delta\omega}{p_1} - \left(1 - \frac{\gamma_2 - 1}{\gamma_2}\frac{\gamma_1}{\gamma_1 - 1}\right)\left(\delta u_1 + \delta S_a\right)_h \tag{13b}$$

Substituting equations (13) into equation (11) gives the important relation

$$\frac{\partial}{\partial t}(E) = A\left(\delta p_1\right)_h\left[\frac{\gamma_2 - 1}{\gamma_2}\frac{\delta\omega}{p_1} - \left(1 - \frac{\gamma_2 - 1}{\gamma_2}\frac{\gamma_1}{\gamma_1 - 1}\right)\left(\delta u_1 + \delta S_a\right)_h\right] \tag{14}$$

Now, if the energy in the disturbance after each cycle of oscillation is examined,[5] the disturbance is said to be encouraged by the heat source if the total energy in the disturbance at the end of the cycle is greater than that at the beginning of the cycle. If the increase of the disturbance energy after each cycle of oscillation is denoted by Δ, the following equation results:

$$\Delta = \frac{\gamma_2 - 1}{\gamma_2} A \int_{\text{Cycle}} \frac{\left(\delta p_1\right)_h\left(\delta\omega\right)}{p_1}\,dt - \left(1 - \frac{\gamma_1}{\gamma_1 - 1}\frac{\gamma_2 - 1}{\gamma_2}\right) A \int_{\text{Cycle}} \left(\delta p_1\right)_h\left(\delta u_1 + \delta S_a\right)_h\,dt \tag{15}$$

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[5]Since the motion is not strictly periodic (i.e., the motion does not repeat itself exactly after a certain interval), the term "cycle of oscillation" must be clarified a little. It can be conveniently defined as the interval between two successive times at which $\left(\delta p_1\right)_h = 0$.

When the various forms of losses in the system are negligible, the condition that a disturbance is amplified by the heat source in the course of time is then

$$\Delta > 0 \tag{16}$$

If $\Delta < 0$, small disturbances in the system are damped by its interaction with the heat source. If $\Delta = 0$, the heat source neither encourages nor discourages any disturbances. (It merely modifies their wave form.)

In the next section, the above results are applied to a few special cases.

APPLICATIONS

The results of the foregoing analysis can be applied immediately to two special cases of practical interest:

(1) When the heat source is a plane heater and there is a current of air through a tube with Mach numbers of the order of magnitude of 0.01 or less.

(2) When the heat source is a flame front.

Plane Heater

In the case of a plane heater the ratios of the specific heat at constant pressure to the specific heat at constant volume of the medium ahead of and behind the heater may be taken as the same (i.e., $\gamma_1 = \gamma_2 = \gamma$). Equation (15) then becomes

$$\Delta = \frac{\gamma - 1}{\gamma} \frac{A}{p_1} \int_{\text{Cycle}} \left(\delta p_1\right)_h (\delta\omega)\ dt \tag{17}$$

Disturbances in the tube will be amplified in the course of time if $\Delta > 0$. This is Rayleigh's criterion in the form first suggested by Putnam and Dennis (ref. 3). It states that if the time integral of the product of the pressure fluctuation and the fluctuation in the rate of heat release over a cycle of oscillation is greater than zero, the disturbance is amplified; if the integral is less than zero, the disturbance is damped; if the integral is zero, the disturbance is neither amplified nor damped by the heater.

Since neither ω_0 nor u_1 enters into equation (17), the above conclusion applies equally well to the case in which $\omega_0 = 0$ and/or $u_1 = 0$, that is, the case where there is no current of air through the tube and/or there is no steady component of heating.

Flame Front

In the case of the flame front, γ_1 is, in general, different from γ_2. Furthermore, from the definition of the flame speed and equation (6d),

$$\delta u_1 + \delta S_a = \delta S_t \tag{18}$$

Hence,

$$\Delta = \frac{\gamma_2 - 1}{\gamma_2} A \int_{\text{Cycle}} \frac{(\delta p_1)_h(\delta\omega)}{p_1} dt - \left(1 - \frac{\gamma_1}{\gamma_1 - 1}\frac{\gamma_2 - 1}{\gamma_2}\right) A \int_{\text{Cycle}} (\delta p_1)_h \delta S_t \, dt \tag{19}$$

This formula for the increase of total energy per cycle of oscillation is strictly true for a <u>plane</u> flame front propagating at a speed of the order of magnitude of 10 feet per second or less. Physically, it seems probable that it may perhaps also be applied to a nonplanar flame front when the axial extent occupied by the curved flame front is small compared with the length of the tube, provided that $\delta\omega$ is interpreted as the change of total heat release per second per cross-sectional area of the tube (see ref. 9). Since the rate at which heat is released by a flame front (plane or nonplanar) is equal to the product of the flame area, the flame speed, the density of the fresh gas, and the heating value of the mixture, any changes (induced by the fluctuation in the system) of any of these variables will cause a change in the rate of heat release per unit time per unit cross-sectional area of the tube. In fact, since $\omega_0 = A_f S_t \bar{Q} \rho_1 / A$, the effect of a change in the flame area A_f and the flame speed on the rate of heat release is, to the order of approximation which has been used in deriving equations (12e) and (9), given by

$$\frac{\delta\omega}{p_1c_1} = \frac{S_t}{c_1}\,\frac{\overline{Q}}{R_1T_1}\,\frac{\delta A_f}{A} + \frac{\delta S_t}{c_1}\,\frac{\overline{Q}}{R_1T_1}\,\frac{A_f}{A} \tag{20}$$

The effect of changes in the heating value on $\delta\omega$ has been ignored here mainly because it is normally very small. Substituting equation (20) into equation (19) gives

$$\Delta = \frac{\gamma_2 - 1}{\gamma_2}\,\frac{\overline{Q}}{R_1T_1}\,S_t \int_{\text{Cycle}} (\delta p_1)_h \delta A_f\,dt + \left(\frac{\rho_1}{\rho_2} - 1\right)A \int_{\text{Cycle}} (\delta p_1)_h \delta S_t\,dt \tag{21}$$

where use has been made of equations (6a), (6c), and (6d) and $\omega_b = A_f S_t \overline{Q} \rho_1 A$ to establish the relation

$$\frac{\gamma_2 - 1}{\gamma_2}\,\frac{\overline{Q}}{R_1T_1}\,\frac{A_f}{A} - 1 + \frac{\gamma_1}{\gamma_1 - 1}\,\frac{\gamma_2 - 1}{\gamma_2} = \frac{\rho_1}{\rho_2} - 1 \tag{22}$$

employed in simplifying the result of the substitution. Consequently, if the pressure fluctuation is positively correlated[6] with the fluctuations in the flame area and flame speed, vibration is encouraged. If they are negatively correlated, vibration is discouraged.

In the study of the stability of systems containing a heat source, the basic problem is then to determine the response of the heat source to the disturbances. Detailed analysis of the dynamic characteristics of the heat source is usually very difficult. The major contribution of Carrier in the study of Rijke tone is precisely in this direction (see ref. 10). When a detailed analysis for a particular system proves to be too complicated (as is usually the case when a flame front is involved), the dynamic response of the heat source can often be assumed if the study is accompanied by experimental investigations. The excellent work of Bailey (ref. 6) is a typical example of this approach and another example can be found in reference 7. Still another example in this connection can be constructed by making use of the experimental data of Kaskan (ref. 3). In his study of oscillatory flame propagation in

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[6]Two variables are said to be positively correlated if the time integral of their product over a cycle of oscillation is greater than zero.

tubes, Kaskan proved beyond any doubt that in such a case the vibration is amplified mainly because of the proper phasing of the change of the flame area with the pressure fluctuation. He went on to propose a mechanism which explains this interaction between the flame front and the pressure waves. Kaskan's picture of the interaction is essentially as follows: When a flame extends itself into the oscillating boundary layer near the wall, the motion of the central portion of the flame front is much greater than that inside the boundary layer. Consequently, there is a change in the flame area whenever there is an acoustical oscillation in the tube. This picture of the interaction is supported further by the experimental evidence that if the oscillatory frequency is so high and, hence, the boundary layer is so thin that the quenching distance of the flame near the wall is greater than the boundary-layer thickness (i.e., the flame does not extend itself into the boundary layer), then there is no change in flame area with any acoustical oscillation inside the tube and there is no evidence that the small disturbances are amplified and that the flame vibrates. The excellent experimental study of Kaskan should be followed up with a theoretical study of this phenomenon. The main question then is to calculate the flame configuration (and, hence, the accompanying rate of heat release) in an oscillating flow inside a tube - much in the same spirit of Carrier in his study of Rijke tone. The difficulty here is that even an approximate calculation of the flame configuration proves to be too complicated. However, a very crude estimation of the rate of heat release at the flame front can be made on the basis of Kaskan's results. Thus, by making the two assumptions (1) The flame configuration remains essentially similar during its vibration so that the flame area at any instant is proportional to the height of the curved flame and (2) the central portion of the flame moves back and forth as if there were no boundary-layer effects while the edge of the flame in the boundary layer only moves forward with a constant velocity equal to the mean speed of propagation of the flame, it is easy to show that the change in flame area is directly proportional to the fluctuation in the displacement of the fluid particle at the flame front. This can serve as a starting point of a theoretical calculation of the oscillatory flame propagation in tubes along a line similar to that Bailey has followed in reference 6.

The Johns Hopkins University,
Baltimore, Md., May 31, 1955.

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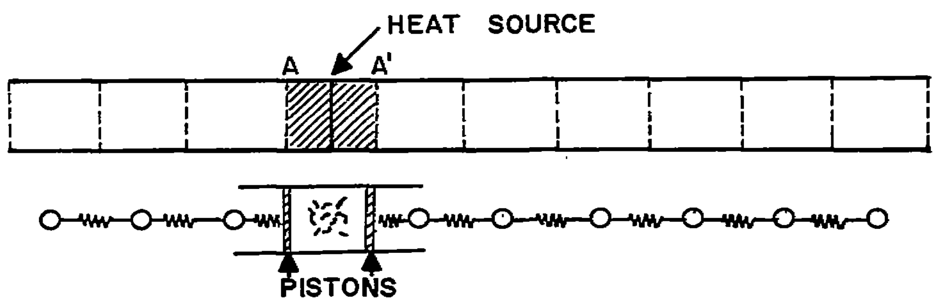

Figure 1.- Sketch showing engine analogy. Dashed lines indicated planes separating compartments.

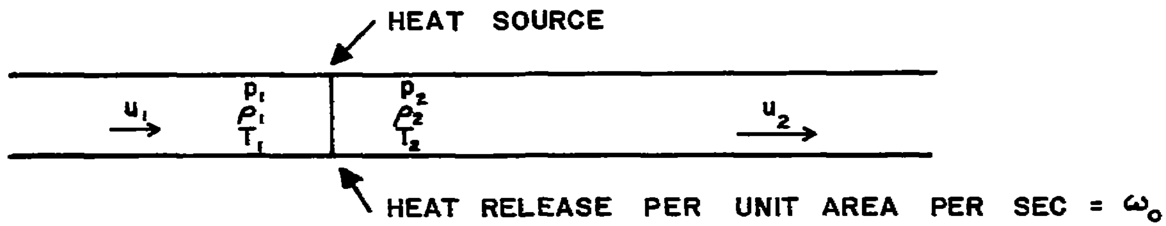

Figure 2.- Heat addition in a flowing stream.

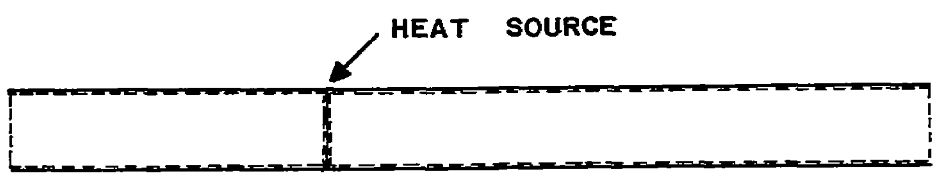

Figure 3.- Control surface shown as dashed line (two compartments).